Math 55 Quiz 3 DIS 105

Name: ______________________________ 14 Feb 2022

1. Prove or disprove each of the following statements:

   (a) The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x + 1$ is onto. [3 points]

   (b) If f is an injective function from A to B, and S and T are subsets of B, then $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$. [4 points]

   (c) The set of positive integers which have remainder 1 when divided by 3 is countable. [3 points]

(a) This is false. We claim that $0 \in \mathbb{N}$ is not mapped to by any element of $\mathbb{N}$:
Suppose otherwise, then $f(x) = x + 1 = 0$ for some $x \in \mathbb{N}$. But then we must have $x = -1$ and so x does not lie in $\mathbb{N}$; contradiction.

(b) This is true.
Suppose $x \in f^{-1}(S \cap T)$, then $f(x) \in S \cap T$, so $f(x) \in S$ and $f(x) \in T$. This implies that $y = f(x) \in f(S)$ and $y = f(x) \in f(T)$, so $y \in f(S) \cap f(T)$. Hence $f^{-1}(S \cap T) \subseteq f^{-1}(S) \cap f^{-1}(T)$.
Conversely, suppose $x \in f^{-1}(S) \cap f^{-1}(T)$, then $x \in f^{-1}(S)$ and $x \in f^{-1}(T)$. $x \in f^{-1}(S)$ implies that $f(x) \in S$, and $x \in f^{-1}(T)$ implies that $f(x) \in T$, so $f(x) \in S \cap T$, hence $x \in f^{-1}(S \cap T)$. Hence $f^{-1}(S) \cap f^{-1}(T) \subseteq f^{-1}(S \cap T)$.
Hence $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

(c) This is true. Let S be the set of positive integers which have remainder 1 when divided by 3. We define a function $f : \mathbb{N} \to S$ by $f(n) = 3n + 1$, and we claim that f is surjective:
For any $k \in S$, since k has remainder 1 when divided by 3, $k = 3n + 1$ for some $n \in \mathbb{Z}$. $n \geq 0$ or else $k = 3n + 1 \leq 3(-1) + 1 = -2$, so $n \in \mathbb{N}$. Hence $f(n) = k$.